THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2060B Mathematical Analysis II (Spring 2017)
Solution to Midterm Examination

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1. Define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) := \begin{cases} x^2, & \text{if } x \in \mathbb{Q} \\ 0, & \text{otherwise} \end{cases}$$

Show that $f'(0)$ exists but $f'(x)$ does not exist for any $x \neq 0$.

Proof.

- We first show that $f'(0)$ exists. (The term x^2 suggests that $f'(0) = 0$). Since $f(0) = 0$, it suffices to show that

$$\lim_{x \to 0} \frac{f(x)}{x} = 0.$$

Let $\epsilon > 0$, and take $\delta := \epsilon$. For any $|x| < \delta$, if $x \in \mathbb{Q}$, then

$$\left| \frac{f(x)}{x} \right| = \left| \frac{x^2}{x} \right| = |x| < \epsilon.$$

If $x \notin \mathbb{Q}$, then

$$\left| \frac{f(x)}{x} \right| = \left| \frac{0}{x} \right| = 0 < \epsilon.$$

This proves that $f'(0) = 0$.

- Next we show that for $x \neq 0$, $f'(x)$ does not exist. It is easier to show that f is not continuous at x, hence not differentiable at x. By the sequential criterion it suffices to exhibit a sequence x_n converging to x but $f(x_n)$ does not converge to $f(x)$. We consider two cases.
  - $x \in \mathbb{Q} \backslash \{0\}$. In this case, $f(x) = x^2 \neq 0$. However, by density of irrational numbers, there exists a sequence $x_n \notin \mathbb{Q}$ so that $x_n \to x$. Thus $f(x_n) = 0 \to 0 \neq f(x)$. Hence f is discontinuous at x.
  - $x \notin \mathbb{Q}$. In this case, $f(x) = 0$. By density of rational numbers, take $x_n \in \mathbb{Q}$ so that $x_n \to x$. Thus $f(x_n) = x_n^2 \to x^2 \neq 0$, since $x \neq 0$. Hence f is discontinuous at x.

To conclude, $f'(x)$ does not exist at $x \neq 0$.

□

2. Suppose f is differentiable on a bounded open interval (a, b).

   (a) Show that if f is unbounded on the interval (a, b), then f' is also unbounded on (a, b).

(b) Does the converse of part (a) hold?

Proof. (a) We prove by contradiction. Suppose f' is bounded on (a,b). We have two different approaches to show that f is bounded on (a,b).

- Method 1: Since f' is bounded on (a,b), let $M := \sup_{x\in(a,b)} |f'(x)| < \infty$. We show that f is Lipschitz continuous. Indeed, given $a < x < y < b$, since f is continuous on $[x,y]$ and differentiable on (x,y), by the mean value theorem, there is some $z \in (x,y)$ so that

$$f(y) - f(x) = f'(z)(y-x).$$

But then

$$|f(y) - f(x)| = |f'(z)(y-x)| \leq M|y-x|,$$

showing that f is Lipschitz continuous on (a,b). Then f is in particular uniformly continuous on (a,b). By the uniform extension theorem, f can be continuously extended to $\tilde{f} : [a,b] \to \mathbb{R}$ so that $\tilde{f}$ is uniformly continuous. In particular, $\tilde{f}$ is bounded on $[a,b]$. In particular, f is bounded on (a,b).

- Method 2: Fix an arbitrary $c \in (a,b)$. For any $x \in (c,b)$, since f is continuous on $[c,x]$ and differentiable on (c,x), by the mean value theorem, there is some $z \in (c,x)$ so that

$$f(x) - f(c) = f'(z)(x-c).$$

But then

$$|f(x) - f(c)| = |f'(z)(x-c)| \leq M|x-c| \leq M(b-a),$$

thus by the triangle inequality, for any $x \in (c,b)$,

$$|f(x)| \leq |f(x) - f(c)| + |f(c)| \leq M(b-a) + |f(c)|.$$

Similarly, for any $x \in (a,c)$, $|f(x)| \leq M(b-a) + |f(c)|$. Therefore for any $x \in (a,b)$, $|f(x)| \leq M(b-a) + |f(c)|$, which is a finite constant. Hence f is bounded on (a,b).

(b) The converse does not hold. Consider $f(x) := \sqrt{x}$ defined on $x \in (0,1)$. Then $f'(x) = \frac{1}{2\sqrt{x}}$ on $(0,1)$, which is unbounded. However, $0 < f(x) < 1$ on $(0,1)$, hence f is bounded.

□

3. Define a function $f : [0,1] \to \mathbb{R}$ by

$$\begin{cases} x, & \text{if } x \in \mathbb{Q} \cap [0,1] \\ -x, & \text{otherwise.} \end{cases}$$

Find the upper and lower integrals of f.

Proof. We claim that $\overline{\int_0^1} f = \frac{1}{2}$ and $\underline{\int_0^1} f = -\frac{1}{2}$:

For any partition $P = \{x_0 = 0, ..., x_N = 1\}$ of $[0,1]$, we first show that for each $1 \leq i \leq N$,

$$\sup_{[x_{i-1},x_i]} f = x_i$$

and similarly

$$\inf_{[x_{i-1},x_i]} f = -x_i$$

For the former one, first by definition of f we immediately see that $f(x) \leq x_i$ for all $x \in [x_{i-1}, x_i]$, and therefore $\sup_{[x_{i-1},x_i]} f \leq x_i$; On the other hand, fix any $\epsilon > 0$, by density theorem of rational numbers, there exists $y_i \in (x_i - \epsilon, x_i) \cap \mathbb{Q}$. Therefore, $x_i - \epsilon < f(y_i) < x_i$. Since $\epsilon > 0$ is arbitrary, we have

$$\sup_{[x_{i-1},x_i]} f = x_i.$$

For the latter one the argument is analogous: by definition of f we immediately see that $f(x) \geq -x_i$ for all $x \in [x_{i-1}, x_i]$, and therefore $\inf_{[x_{i-1},x_i]} f \geq -x_i$; On the other hand, fix any $\epsilon > 0$, by density theorem of irrational numbers, there exists $z_i \in (x_i - \epsilon, x_i) \cap (\mathbb{R} - \mathbb{Q})$. Therefore, $-x_i < f(z_i) < -x_i + \epsilon$. Since $\epsilon > 0$ is arbitrary, we have

$$\inf_{[x_{i-1},x_i]} f = -x_i.$$

Therefore, $U(f,P) = \sum_{i=1}^N x_i(x_i - x_{i-1}) = U(g,P)$ and $L(f,P) = \sum_{i=1}^N (-x_i)(x_i - x_{i-1}) = L(h,P)$, where $g, h : [0,1] \to \mathbb{R}$ is given by $g(x) = x$ and $h(x) = -x$.

Since g, h are continuous, they are Riemann integrable, i.e. $\overline{\int_0^1} g = \int_0^1 x dx = \frac{1}{2}$ and $\underline{\int_0^1} h = \int_0^1 (-x) dx = -\frac{1}{2}$.

Finally, we compute $\overline{\int_0^1} f$ and $\underline{\int_0^1} f$:

$$\overline{\int_0^1} f = \inf_P U(f,P) = \inf_P U(g,P) = \overline{\int_0^1} g = \frac{1}{2}$$

$$\underline{\int_0^1} f = \sup_P L(f,P) = \sup_P L(h,P) = \underline{\int_0^1} h = -\frac{1}{2}$$

□

4. Define a function f on $[0,1]$ by

$$f(x) := \begin{cases} 1, & \text{if } x = \frac{1}{n}, n = 1, 2, \ldots \\ 0, & \text{otherwise .} \end{cases}$$

Show that f is Riemann integrable and find $\int_0^1 f$.

Proof. It is reasonable to guess that $\int_0^1 f = 0$. Hence it suffices to show: for any $\epsilon > 0$, there exists a partition P such that $U(f,P) - L(f,P) < \epsilon$. But it is easy to see that $L(f,P) = 0$ for any partition P. Hence it suffices to show that $U(f,P) < \epsilon$, which also implies that $\int_0^1 f = 0$.

Let $\epsilon > 0$. Assume $\epsilon < 0.1$ without loss of generality. Let N be the least integer such that $N > \frac{1}{\epsilon}$. Note then $N \geq 10$. Consider the points $\frac{1}{k}, k = 1, 2, \ldots, N-1$, and take

$$\delta := \min\left\{\frac{1}{N-2} - \frac{1}{N-1}, \frac{1}{N-1} - \frac{\epsilon}{2}\right\} \in \left(0, \frac{2}{N(N-1)}\right).$$

Consider the partition P given by:

$$0 = x_0 < \frac{\epsilon}{2} < \frac{1}{N-1} - \frac{\delta}{100} < \frac{1}{N-1} + \frac{\delta}{100} < \frac{1}{N-2} - \frac{\delta}{100} < \frac{1}{N-2} + \frac{\delta}{100}$$
$$< \cdots < \frac{1}{2} + \frac{\delta}{100} < 1 - \frac{\delta}{100} < 1 = x_n.$$

Then we can compute

$$\begin{aligned}
U(f,P) &= \sum_{i=1}^{n}(x_i - x_{i-1}) \sup_{x\in[x_{i-1},x_i]} f(x) \\
&= (x_1 - x_0) \sup_{x\in[0,\frac{\epsilon}{2}]} f(x) + \sum_{i=2}^{n}(x_i - x_{i-1}) \sup_{x\in[x_{i-1},x_i]} f(x) \\
&\leq \frac{\epsilon}{2} \cdot 1 + (N-1) \cdot \frac{\delta}{50} \cdot 1 \\
&= \frac{\epsilon}{2} + \frac{(N-1)\delta}{50} \\
&< \frac{\epsilon}{2} + \frac{1}{25N} \\
&< \epsilon.
\end{aligned}$$

□